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ON SUFFICIENT OPTIMALITY CONDITIONS

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Sufficient conditions of optimality of the control in a nonlinear system are given. This involves a demand for existence of a function with specified properties. If this function is defined in a special manner, then the theorem derived in the paper yields the known theorem of Krotov [1]. A certain relaxation of the sufficient conditions given in [1] is obtained for the problems of the time optimal response in autonomous systems.

1. Let the controlled object be characterized by the phase coordinates $x = (x^1, x^2, \ldots, x^n)$ in an n-dimensional Euclidean space E^n the law of variation of which is described by the differential equation

$$dx/dt = f(x, u, t) \tag{1.1}$$
$$(u = (u^1, u^2, \ldots, u^r),\ f = (f^1, f^2, \ldots, f^n))$$

where u is an r-dimensional control vector. The components of the vector function $f(x, u, t)$ are assumed to be continuous in all its arguments, and continuously differentiable with respect to the variables x^i, $i = 1,2, \ldots, n$. We adopt, as the admissible controls, the set of all measurable functions $u(t)$, $t_0 \leqslant t \leqslant t_1$ the values of which satisfy the restriction $u \in U$ where U is a compact in E^r.

Let Ω_0 and Ω_1 represent some admissible closed sets in E^n, and Ω an open set. The time instants t_0 and t_1 are not fixed. We set $t_0 \in T_0 = [\tau_0, \tau_0']$, $t_1 \in T_1 = [\tau_1, \tau_1']$.

The problem of optimal control consists of finding, from amongst all admissible controls which transport the object (1.1) from the position $x_0 \in \Omega_0$ to the position $x_1 \in \Omega_1$, such a control $u(t)$, $t_0 \leqslant t \leqslant t_1$ and the corresponding trajectory $x(t)$, $x(t) \in \Omega$, $t_0 \leqslant t \leqslant t_1$, $x(t_0) = x_0$, $x(t_1) = x_1$, which together impart the possible minimum value to the functional

$$I = \int_{t_0}^{t_1} f^\circ(x, u, t)\, dt$$

The function $f^\circ(x, u, t)$ is assumed to satisfy the same condition as the components of the vector function $f(x, u, t)$.

Let the continuously differentiable function $\varphi(x^\circ, x, t)$ of $n+2$ variables x°, $x^1, x^2, \ldots, x^n, t$ be given. We introduce the function and the sets

$$R(x^\circ, x, u, t) = \frac{\partial\varphi}{\partial x^\circ} f^\circ + \frac{\partial\varphi}{\partial x} f + \frac{\partial\varphi}{\partial t}$$
$$Q = E^1 \times \Omega \times [\tau_0, \tau_1']$$
$$\Pi = \{(x^\circ, x, t)\colon \varphi(x^\circ, x, t) \geqslant 0,\ (x^\circ, x, t) \in Q\}$$

Theorem 1. The sufficient condition for the process $\{x_*(t),\ u_*(t)\}$, $x_*(t) \in \Omega$, $\{x_*(t_0^*),\ t_0^*\} \in \Omega_0 \times T_0$, $\{x_*(t_1^*),\ t_1^*\} \in \Omega_1 \times T_1$ to be optimal is, that a function $\varphi(x^\circ, x, t)$ continuously differentiable on the set Q exists such, that the following conditions hold:

$$\text{A)} \max_{(x,\, t)\in\Omega_0\times T_0} \varphi(0, x, t) = \varphi(0, x_*(t_0^*), t_0^*) = 0$$

$$\text{B)} \sup_{u\in U,\, (x^\circ, x, t)\in\Pi} R(x^\circ, x, u, t) \leqslant 0$$

$$R(I_*(t), x_*(t), u_*(t), t) = 0, \quad t_0^* \leqslant t \leqslant t_1^*$$

$$\text{C)}\ \varphi(\xi, x, t) > 0, \quad x \in \Omega_1, \quad t \in T_1, \quad \xi < I_*(t_1^*)$$

where

$$I_*(t) = \int_{t_0^*}^{t} f^\circ(x_*(t), u_*(t), t)\, dt$$

Proof. Let us consider the following system of differential equations in the space E^{n+2} :

$$\frac{dx^\circ}{dt} = f^\circ(x, u, t), \quad \frac{dx}{dt} = f(x, u, t), \quad \frac{dx^{n+1}}{dt} = 1 \tag{1.2}$$

By choosing an arbitary admissible control $u(t)$, $t_0 \leqslant t \leqslant t_1$ and the initial Cauchy conditions

$$x^\circ(t_0) = 0,\ x(t_0) = x_0 \in \Omega_0, \quad x^{n+1}(t_0) = t_0 \in T_0 \tag{1.3}$$

we define a trajectory

$$x^\circ(t),\ x(t),\ x^{n+1}(t) \equiv t,\ t_0 \leqslant t \leqslant t_1 \tag{1.4}$$

of the system (1.2). The equation

$$\varphi(x^\circ, x, t) = 0 \tag{1.5}$$

separates the set Q into two subsets. Let us denote by Q^+ the subset of Q on which the function $\varphi(x^\circ, x, t)$ is positive, and by Q^- the other subset. Condition (A) implies that the initial set (1.3) is completely contained in Q^- and the point $(0, x_*(t_0^*), t_0^*)$ lies on the surface (1.4). Condition (B) implies that the surface (1.5) is "impermeable", i.e. the trajectory of the system (1.2) emerging from the set (1.3) will remain within Q^- , under any admissible control $u(t)$, $t_0 \leqslant t \leqslant t_1$, $t_0 \in T_0$, $t_1 \in T_1$, during the whole process. At the same time, the integral curve

$$(x_*^0(t) = I_*(t)), \quad x_*(t), \quad x_*^{n+1}(t) = t, \quad t_0^* \leqslant t \leqslant t_1^*$$

lies on the surface (1.5), i.e.

$$\varphi(I_*(t), x_*(t), t) = 0, \quad t_0^* \leqslant t \leqslant t_1^* \tag{1.6}$$

Let us assume that the process in question is not optimal, i.e. that there exists a process $\{x(t), u(t)\}$, $t_0 \leqslant t \leqslant t_1$, $x(t) \subset \Omega$, $\{x(t_0), t_0\} \in \Omega_0 \times T_0$, $\{x(t_1), t_1\} \in \Omega_1 \times T_1$, such, that

$$I < I_*(t_1^*) \tag{1.7}$$

Consider the integral curve (1.4) of the system (1.2). Since $x^\circ(t_0) = 0$ $x(t_0) \in \Omega_0$, $t_0 \in T_0$, and $u(t)$, $t_0 \leqslant t \leqslant t_1$ is an admissible control, the integral curve lies, as we showed before, in the subset Q^-, i.e.

$$\varphi(x^\circ(t), x(t), t) \leqslant 0, \quad t_0 \leqslant t \leqslant t_1$$

But condition (C) and the inequality (1.7) together imply that

$$\varphi(x^\circ(t_1), x(t_1), t_1) > 0$$

and the resulting contradiction proves the theorem.

If the process $\{x_*(t), u_*(t)\}$ satisfies the condition of Theorem 1, then we have the following inequality:

$$\varphi(I_*(t_1^*), x, t) \geqslant 0, \; x \in \Omega_1, \; t \in T_1 \tag{1.8}$$

Indeed, let the opposite inequality hold at some point $x = a \in \Omega_1$ and $t = \mu \in T_1$:

$$\varphi(I_*(t_1^*), a, \mu) = b < 0$$

From condition (C) we have, for any $\varepsilon > 0$,

$$\varphi(I_*(t_1^*) - \varepsilon, a, \mu) = b - \frac{\partial \varphi(I_*(t_1^*), a, \mu)}{\partial x^\circ}\varepsilon + o(\varepsilon) > 0$$

and this is impossible, since $b < 0$ by definition.

Finally we note, that the inequality (1.8) becomes an equality at the point $x_*(t_1^*) \in \Omega_1$, $t_1^* \in T_1$. This follows directly from (1.6) at $t = t_1^*$.

All this, makes possible the following assertion:

$$\min_{(x,t) \in \Omega_1 \times T_1} \varphi(I_*(t_1^*), x, t) = 0$$

The above expresssion formally coincides with condition (A) of Theorem 1; it is not however equivalent to condition (C), being substantially weaker.

If we define the function $\varphi(x^\circ, x, t)$ in the following form:

$$\varphi(x^\circ, x, t) = K(x, t) - x^\circ \tag{1.9}$$

then a theorem due to Krotov [1] follows from Theorem 1.
Theorem 1 given above ans stating the sufficient conditions of optimality, is a direct generalization of the results of [3].

2. Let the behavior of the object be described by

$$x^\cdot = f(x, u)$$

Consider the problem of fast response when $\Omega_0 = \{x_0\}$, $\Omega_1 = \{x_1\}$. Let $\{x(t), u(t)\}$, $0 \leqslant t \leqslant t_1$ be a process satisfying the Pontriagin maximum principle [4], and $\psi(t)$, $0 \leqslant t \leqslant t_1$ be a vector function corresponding to this process. Let us set

$$c(\psi, x) = \max_{u \in U} (\psi, f(x(u))$$

Then provided that the control $u(t)$, $0 \leqslant t \leqslant t_1$ is a piecewise continuous function, the following corollary can be obtained from Theorem 1.

T h e o r e m 2. Let the function $c(\psi, x)$ be such that

$$c(\psi(t), x) - c(\psi(t), x(t)) - \left(\frac{\partial c(\psi(t), x(t))}{\partial x}, x - x(t)\right) \leqslant 0 \tag{2.1}$$

when $(\psi(t), x - x(t)) \geqslant 0$, and let the following condition hold:

$$(\psi(t), x_1 - x(t)) > 0, \ 0 \leqslant t < t_1 \tag{2.2}$$

Then the process $\{x(t), u(t)\}$, $0 \leqslant t \leqslant t_1$ is optimal with respect to the time optimal response.

P r o o f. To apply Theorem 1 to the problem of time optimal response we must put $f^\circ \equiv 1$ and use the time t as the coordinate x°. Then, instead of the function $\varphi(x^\circ, x, t)$ we shall have $\varphi_1(t, x)$ and instead of $R(x^\circ, x, u, t)$, the function

$$R_1(t, x, u) = \frac{\partial \varphi_1}{\partial t} + \frac{\partial \varphi_1}{\partial x} f(x, u)$$

and the following sets, respectively

$$Q_1 = [0, t_1] \times E^n, \Pi_1 = \{(t, x): \varphi_1(t, x) \geqslant 0, (t, x) \in Q_1\}$$

For the process $\{x(t), u(t)\}$ to be optimal, it is sufficient that a function $\varphi_1(t, x)$, continuously differentiable on the set Q_1 exists such that the following conditions hold:

$A_1)$ $\varphi_1(0, x(0)) = 0$

$B_1)$ $\sup\limits_{u \in U, (t, x) \in \Pi_1} R_1(t, x, u) \leqslant 0$

$C_1)$ $\varphi_1(t, x_1) > 0, \quad t < t_1$

Let us set

$$\varphi_1(t, x) = (\psi(t), x - x(t)) \tag{2.3}$$

The above function is continuously differentiable everywhere on the set Q_1 except at the points of a finite number of planes $t = \tau_i, i = 1,2, \ldots, N$ where τ_i denote points on the segment $[0, t_1]$ at which the function $u(t)$ has first order discontinuities. We note that Theorem 1 remains valid when the function ceases to be continuously differentiable at the points of a finite number of planes

$$t = \tau_i \ (i = 1,2, \ldots, N), \quad \tau_i = \text{const}$$

When the function $\varphi_1(t, x)$ is given by (2.3), condition (A_1) is fulfilled automatically and condition (B_1) assumes the form

$$\sup_{u \in U} R_1(t, x, u) \leqslant 0; \quad 0 \leqslant t \leqslant t_1, \quad (\psi(t), x - x(t)) \geqslant 0 \tag{2.4}$$

Let us transform the left-hand side of the inequality (2.4), with the particular form (2.3) of the function $\varphi_1(t, x)$ taken into account. We have

$$\frac{\partial \varphi_1(t, x)}{\partial x} = \psi(t), \quad \frac{\partial \varphi_1(t, x)}{\partial t} = (\psi^\cdot(t), x - x(t)) - \tag{2.5}$$

$$(\psi(t), x^\cdot(t)) = -\left(\frac{\partial c(\psi(t), x(t))}{\partial x}, x - x(t)\right) - c(\psi(t), x(t))$$

Here we have used the Pontriagin maximum principle

$$(\psi(t), f(x(t), u(t)) = \max_{u \in U} (\psi(t), f(x(t), u)) = c(\psi(t), x(t))$$

and the relation [2] $\psi^{\cdot}(t) = -\partial c(\psi(t), x(t))/\partial x$, which holds under the assumption that the function $c(\psi, x)$ is differentiable.

Using the relations (2.5) we conclude, that $R_1(t, x, u) \leqslant Q(t, x)$ where $Q(t, x)$ is the left-hand side of the inequality (2.1). Therefore the condition (2.1) guarantees the validity of condition (B_1), and condition (C_1) can be written in the form (2.2), which completes the proof of Theorem 2.

In the theorem given in [2] the inequality (2.1) was required to hold over the whole space E^n.

3. It can be seen from the formula (1.9) that the sufficient conditions of optimality due to Krotov [1] follow from Theorem 1 provided that the equation $\varphi(x^\circ, x, t) = 0$ can be solved for x°, i.e. the inequality

$$\partial\varphi(x^\circ, x, t)/\partial x^\circ \neq 0 \tag{3.1}$$

must hold over the whole domain of variation of the variables $(x, t) \in \Omega \times [\tau_0, \tau_1']$.

Let us assume that $f^\circ(x, u, t) > 0$ for all $u \in U$, $x \in \Omega$, $t \in [\tau_0, \tau_1']$. Let $\{x_*(t), u_*(t), \psi^*(t)\}$, $t_0 \leqslant t \leqslant t_1$ be the Pontriagin extremal in the problem of Sect. 1. The equation

$$x^\circ = \int_{t_0}^{t} f^\circ(x_*(\tau), u_*(\tau), \tau)\, d\tau$$

defines uniquely the function $t = \xi(x^\circ)$ by virtue of the assumption made with respect to the function $f^\circ(x, u, t)$. As in Sect. 2 setting

$$\varphi(x^\circ, x, t) = (\psi^*(\xi(x^\circ)),\ x - x_*(\xi(x^\circ))) + \psi^*_{n+1}(\xi(x^\circ))(t - \xi(x^\circ))$$

we can obtain the sufficient conditions of optimality for the extremal $\{x_*(t), u_*(t), \psi^*(t)\}$, $t_0 \leqslant t \leqslant t_1$, similar to those formulated in Theorem 2. The equation

$$(\psi^*(\xi(x^\circ)), x - x_*(\xi(x^\circ))) + \psi^*_{n+1}(\xi(x^\circ))(t - \xi(x^\circ)) = 0$$

is not solved for x° and, in addition, the condition (3.1) does not hold except in the trivial cases.

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